Increasing Returns and Competitive Equilibrium

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By Erdem Basci and Ismail Saglam[1]

This paper is about the viability of competitive equilibrium under increasing returns to scale technologies in a finite-horizon production economy that operates with the help of money. We argue that presence of cash-in-advance constraints in input markets eliminates the problem of unbounded factor demands. We introduce some restrictions on utility and production functions to ensure that the first-order necessary condition (Euler equation), for the optimization problem of a producer with non-concave objective function, is also sufficient. We then derive a condition on the growth rate of money which is both necessary and sufficient for the existence and uniqueness of a stationary monetary competitive equilibrium. We fully characterize the equilibrium and provide a graphical procedure for conducting comparative statics. The effects of changes in money growth rate, number of firms, labor endowment and productivity on equilibrum prices and allocations are analyzed. We observe that Friedman's rule (or output maximizing money growth rule) is not necessarily deflationary. Furthermore we show that if producers are sufficiently patient, the optimal money growth rule implements the average cost pricing equilibrium, without any need for direct price regulation.

Keywords: Friedman's rule, increasing returns, competitive equilibrium, money.

1. INTRODUCTION

Convexity is a key assumption in general competitive analysis. In particular, convexity of production sets, or equivalently concavity of production

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functions, is assumed and used in existence proofs as well as in welfare theorems of Walrasian general equilibrium theory (Arrow and Debreu (1954), Debreu (1959)). In this paper, we argue and demonstrate by means of a simple dynamic model that in monetary economies competitive equilibrium may exist even under the severest form of non-convexity in production, namely increasing returns to scale technologies.

There are two obstacles against the operation of the price system under increasing returns technologies. The first is due to the unbounded input demands and output supplies for any given non-zero output prices. This is because of increasing marginal profitability from expanding the productive activity. The second obstacle becomes relevant for regulated production plants with quotas on the input or the output side. This obstacle is the incentive compatibility problem, as observed in the marginal cost pricing equilibrium which requires economic losses waiting to be compensated by subsidies from the government, or in the average cost pricing equilibrium where the firm would like to expand output under the prices announced by the regulatory body. The normative properties of marginal and average cost pricing equilibria are well understood in the economics literature.[2] Nevertheless, it is also recognized that the informational problems of the regulator together with the incentive problems of the regulated firms is likely to give rise to inefficient outcomes, deadweight losses or informational corruption.[3]

In the present paper, we first ask whether unregulated competitive equilibrium may exist in an economy with a finite number of firms, each endowed with an increasing returns technology. We eliminate the first obstacle of unbounded input demands by letting the economy to operate with fiat money and letting the firms use money as working capital in financing of

[2]For example, Beato (1982), Brown and Heal (1983), Guesnerie (1975), Khan and Vohra (1987) and Vohra (1988, 1992).

[3]See, for example, Baron and Myerson (1982) that introduces the optimal Bayesian incentive mechanism to regulate a natural monopoly, and Saglam (1997) that examines corruption and learning in Bayesian regulatory mechanisms.

their operations. This approach is in line with a recently introduced class of macroeconomic models, with considerable empirical support from data, which assume the payments for inputs have to be made before sales revenue is collected.[4] We eliminate the second obstacle by allowing the market system to operate in a decentralized fashion. This approach eliminates the incentive compatibility problem, in our monetary economy. Nevertheless the welfare properties turn out to depend crucially on the money growth rule used by the monetary authorities.

As required by the competitive equilibrium definition, the choices of our unregulated firms should be consistent with maximization of owners' welfare. However, the customarily used Euler equations are not sufficient for a maximum in our setup with increasing returns. In order to overcome this technical problem, we take an indirect route and use the necessity of Euler equations together with conditions to ensure the existence and uniqueness of solutions to the Euler equations and the existence of a maximum.

After characterizing equilibrium, we analyze its welfare properties. In addition to comparative statics on the number of firms, number of workers, patience levels of firms and workers, in particular, we analyze the optimal money growth rule. Inflationary monetary policies are observed to reduce employment and output, in line with the formerly analyzed constant and decreasing returns cases. The optimal money supply rule (Friedman's rule) under increasing returns, however, differs from the rule that optimally leads to marginal cost pricing under decreasing returns.[5] Under increasing returns,

[4]Fuerst (1992), Carlstrom and Fuerst (1995) and Christiano, Eichenbaum and Evans (1997, 1998, 2001) are some examples. Barth and Ramey (2001) provides empirical support for the relevance of such models at the industry and aggregate levels.

[5]Friedman (1969) argues that an inflation rate that maximize real cash holdings would implement social optimum. Friedman himself and the following microfounded studies like those of Bewley (1980), Cooley and Hansen (1989), Woodford (1990) and Wu and Zhang (1998) assert that this would correspond to a deflation that makes nominal interest rates zero. Practicioners of monetary policy, other than the recent Japanese case, however, try to avoid such an outcome without much theoretical basis but with a strong aversion to

marginal cost pricing is not feasible in a decentralized setup. If average cost pricing is considered to be a second best alternative, we show that such a social optimum can be attained through a money supply rule that yields minimum feasible inflation rate. It is also interesting that the optimal money supply rule corresponding to average cost pricing is not necessarily deflationary.

Our contribution to the literature is twofold. First, we show that unregulated competitive equilibria may exist in economies using money as working capital. Second, we show that an average pricing equilibrium can be attained in such an economy by a careful low inflation policy alone without any need for regulation. The policy makers, however, should be careful enough not to set the inflation rate too low, especially so if increasing returns are rather strong in that particular economy.[6]

The paper is organized as follows. Section 2 introduces the model. Section 3 proves the existence and uniquenes of a stationary monetary competitive equilibrium and characterizes the equilibrium. Section 4 analyzes the equilibrium and Section 5 concludes with some remarks.

2. MODEL

2.1. Environment

We consider a production economy described as follows.

Time Horizon: Economy lasts for two periods. Periods are indexed by $t \in \{1, 2\}$.

Agents: There are two types of agents indexed by $i \in \{1, 2\}$. There exist $N_i > 0$ identical agents of type i. Type 1 and 2 will be called as 'worker' and 'producer', respectively.

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episodes of unemployment crises observed during deflationary periods.

[6] Ata and Basci (2001) is the closest paper to ours. But there money, that is backed by commodities, is not fiat, does not grow; the utility function is logarithmic, the production function is quadratic and no welfare analysis for monetary policy is conducted.

Commodities: There are two commodities in each period: a factor of production, labor, and a nonstorable consumption good, apple.

Factor Endowments: Each worker has a labor endowment $\bar{L}_1 > 0$.

Production Technologies: Each producer owns a production technology, represented by the function f_2, to convert labor into apples. We assume that the technology has increasing returns to scale (IRTS), as $f_2(L), f_2'(L), f_2''(L) > 0$ for all $L > 0$.

Utilities: For a type i agent, $c_{i,t}$ denotes the consumption in period t, β_i the time discount, U_i the instantaneous utility function. Workers value leisure.

A representative worker's life-time utility is $\sum_{t=1}^{2} \beta_1^{t-1} U_1(c_{1,t} + v_1(e_{1,t}))$ where $v_1(e_{1,t})$ is his valuation, measured in consumption good, of the leisure $e_{1,t}$.

On the other side, a representative producer's life-time utility is $\sum_{t=1}^{2} \beta_2^{t-1} U_2(c_{2,t})$. We assume that U_1, U_2 and v_1 are twice continuously differentiable, increasing and strictly concave. We also assume:

A0. $v_1'(0) = \infty$ and $v_1'(\bar{L}_1) = 0$.

A1. $U_2'(0) = \infty$.

A2. U_2' satisfies multiplicative separability, i.e. for any $x_1, x_2 \in I\!R_+$, $U_2'(x_1 x_2) = U_2'(x_1)U_2'(x_2)$.

A3. $f_2(x)/g(f_2'(x))$ is increasing in x, where g is the inverse function of $1/U_2'$.

Assumption $A0$ guarantees a positively sloped labor supply curve. Assumptions $A1$ and $A2$ help to show that an interior solution to the optimization problem of producers exists and corner solutions are not optimal. Assumption $A3$ ensures that the afore mentioned interior solution is unique.

Note that IRTS production functions $f_2(L) = \theta L^\eta$ together with the CRRA utility functions $U_2(c) = c^\gamma/\gamma$, where $\theta > 0$, $\eta > 1$ and $\gamma \in [0, 1/\eta)$, satisfy assumptions $A1 - A3$. In the light of these special classes of technologies and utilities, assumption $A3$ can be interpreted as that for a given convex production technology, utility function of producers must be sufficiently concave (or sufficiently close to a logarithmic function).

Money and the Government: In each period, there exists a positive quantity of fiat money in the economy. Let M_{t-1} denote the aggregate money stock at the beginning of period t. Thereon, the economy starts with $M_0 > 0$.

Let $M_{i,t-1}$ be the money holding of each type i agent at the beginning of period t. Then, $M_{i,t}$ is the end-of-period t money balance of a type i agent. Each newborn type i agent is endowed with $M_{i,0}$ units of currency, a (temporary) gift from the government. We assume that the whole money is initially owned by producers. So, $M_{1,0} = 0$ and $M_{2,0} = M_0/N_2$.

Total money stock evolves over time according to the relation

$$M_t = (1 + \alpha)M_{t-1}, \quad t = 1, 2$$

where $\alpha > -1$. The net growth rate and the law of motion of the money supply are common knowledge. The targeted money growth (contraction) comes about through lump-sum money transfers (taxes) distributed across agents in proportion to their beginning-of-life money endowments. Each type i agent, thus, receives $X_{i,t} = (1 + \alpha)^{t-1}\alpha M_{i,0}$ units of currency transfer at the beginning of period t. Since $M_{1,0} = 0$, it is, indeed, only producers who receive (pay) money transfers (taxes).

Money is a *tax backed* currency. However, taxes are not on earned but granted money. It is publicly known that the government will charge, at the end of period 2, to each type i agent a money tax, τ_i, just equal to the total money transfer he has received during his life time. That is, $\tau_i = (1+\alpha)^2 M_{i,0}$. Obviously, $\tau_1 = 0$ and $\tau_2 = M_2/N_2$. Through this tax schedule, the pre-tax total money stock in the economy, M_2, is fully transferred to the government just before agents leave the two-period economy.

The following table summarizes the asymmetry in what the two types of agents are faced with:

	Type 1 Agent (Worker)	Type 2 Agent (Producer)
Labor Endowment	$\bar{L}_1 > 0$	$\bar{L}_2 = 0$
Valuation of Leisure	v_1 (Concave)	$v_2 = 0$
Production Technology	$f_1 = 0$	$f_2 \in \{IRTS\}$
Initial Money Endowment	$M_{1,0} = 0$	$M_{2,0} = M_0/N_2$
Period$-t$ Money Transfer (Tax)	$X_{1,t} = 0$	$X_{2,t} = \alpha(1+\alpha)^{t-1}M_{2,0}$
Terminal Money Tax	$\tau_1 = 0$	$\tau_2 = M_2/N_2$

Table 1.

Society: We denote and describe a *society* by $\mathcal{S} = \{N_i, \bar{L}_i, U_i, v_i, \beta_i, f_i, M_{i,0}\}_{i=1}^{2}$, where all the parameters and variables obey the stated assumptions above.

2.2. Trade Institution

A trade institution for a given society is the description of choice variables for each type of agents, price variables, constraints on the given choice variables determined by the given prices, and a feasibility requirement for the collective choices of agents. The choice variables and prices together with some notational conventions are listed below.

Choice variables of type i agents in period t:

$c_{i,t}$: consumption
$L_{i,t}$: labor demand; (+): demand, (-): supply
$q_{i,t}$: apple demand; (+): demand, (-): supply
$M_{i,t}$: end-of-period t money balance

Money prices in period t:

w_t : nominal wage rate,

p_t : money price per unit of good.

We can now describe market transactions and the problem of each type.

Transactions: There exist three markets: the money, labor and good (apple) markets. The price of the money is taken as numeraire for all times. So, all prices are in terms of money. We also assume that the labor market opens before the good market. The timing of transactions in period t can be described as follows:

• Each agent of type i starts the period with a post-transfer money balance of $M_{i,t-1} + X_{i,t}$.

• Labor market opens. Factor trade, $L_{i,t}$, of each type i agent occurs at the nominal wage rate w_t. All wage bills are paid before the good market opens.

• Apple production takes place with the purchased labor.

• Good market opens. Each type i agent enters the market with $M_{i,t-1} + X_{i,t} - w_t L_{i,t}$ units of currency. Apple is sold at the nominal price p_t. The end-of-period t money balance of type i agent is $M_{i,t} = M_{i,t-1} + X_{i,t} - w_t L_{i,t} - p_t q_{i,t}$.

Agents' Problems: Given the endowment structure described above and a sequence of strictly positive prices $\{w_t, p_t\}_{t=1}^{2}$, a representative agent of type i faces the following problem (P_i):

$$(P_i) \qquad \max \sum_{t=1}^{2} \beta_i^{t-1}\, U_i(c_{i,t} + v_i(\bar{L}_i + L_{i,t})) \;\; \text{subject to for all } t$$

$$(1) \qquad c_{i,t} = f_i(\bar{L}_i + L_{i,t}) + q_{i,t}$$

$$(2) \qquad -\bar{L}_i \leqslant L_{i,t} \leqslant \frac{M_{i,t-1} + X_{i,t}}{w_t}$$

$$(3) \qquad -f_i(\bar{L}_i + L_{i,t}) \leqslant q_{i,t} \leqslant \frac{M_{i,t-1} + X_{i,t} - w_t L_{i,t}}{p_t}$$

$$(4) \qquad M_{i,t} = M_{i,t-1} + X_{i,t} - w_t L_{i,t} - p_t q_{i,t}$$

$$(5) \qquad M_{i,2} \geqslant \tau_i$$

$$(6) \qquad M_{i,0} \text{ is given.}$$

Equation (1) states that consumption in each period is the sum of the home production (applying to type 2 agents, only) and the apple purchases (sales, if negative). The upper bound on labor purchases in (2) comes from the cash-in-advance requirement and the fact that the labor market opens first. The lower bound, when multiplied by -1, shows the maximum amount of labor that can be supplied, which is zero for type 2 agents by assumption. The constraint (3) on apple purchases can be similarly read, taking into account that the payments or receipts in the goods market come after those in the labor market. Constraint (4) describes the cash flow across successive periods. Inequality (5) ensures that each agent is able to pay his money debt to the government just before the economy terminates.

The constraints of (P_i) altogether describe the missing part of our trade institution. We call this institution *financially constrained* by virtue of the fact that a producer is restricted in its labor purchases with the amount of money he holds at the beginning of each period. This financial restriction limits the participation of producers in both factor and output markets, despite the presence of positive marginal profitability.[7] By a *financially constrained production economy* we mean a society $\mathcal{S}$ operating under a financially constrained trade institution, and denote it by $\mathcal{FCE}$.

2.3. Monetary Equilibrium

DEFINITION: The set of sequences $\{p_t, w_t, c_{i,t}, L_{i,t}, q_{i,t}, M_{i,t} \,|\, i = 1,2\}_{t=1}^{2}$ is a *stationary monetary competitive equilibrium* (SMCE) of the financially constrained production economy $\mathcal{FCE}$, if $w_t, p_t > 0$ for all t, and

[7] The term *limited participation* is used in this sense by Christiano, Eichenbaum and Evans (1997).

(i) $\{c_{i,t}, L_{i,t}, q_{i,t}, M_{i,t}\}_{t=1}^{2}$ solves (P_i) for each i, under the sequence $\{w_t, p_t\}_{t=1}^{2}$,

(ii) $N_1 L_{1,t} + N_2 L_{2,t} = 0$, for all t,

(iii) $N_1 q_{1,t} + N_2 q_{2,t} = 0$, for all t,

(iv) $N_1 M_{1,t} + N_2 M_{2,t} = M_t$, for all t.

(v) $c_{i,2}/c_{i,1} = L_{i,2}/L_{i,1} = q_{i,2}/q_{i,1} = 1$ for all i,

(vi) $w_2/w_1 = p_2/p_1 = M_{i,1}/M_{i,0} = M_{i,2}/M_{i,1} = 1 + \alpha$ for all i.

The first condition is life-time utility maximization under perfect foresight of future prices and price taking behavior. The second to fourth conditions state the equilibrium in the labor, good and money markets. The last two conditions are the stationarity of the real and nominal (rescaled w.r.t. money inflation) variables, respectively.

3. STATIONARY MONETARY COMPETITIVE EQUILIBRIUM

In this section we provide a characterization of the stationary monetary competitive equilibrium of the financially constrained economy described in Section 2 above.

3.1. Solving the Model

We shall first obtain the reduced form problem (P_i') of each type i agent. We can eliminate $c_{i,t}$ and $q_{i,t}$ from (P_i), using the equality constraints (1) and (4). Then, we can restrict ourselves to the clearing of the labor and money markets alone, since the good market will automatically clear as well, thanks to a version of Walras' law applicable to our case.

The reduced form problem (P_1') of each worker is

$$(P_1') \qquad \max_{\{L_{1,t}\},\{M_{1,t}\}} \sum_{t=1}^{2} \beta_1^{t-1}\, U_1\left(-\frac{w_t}{p_t}L_{1,t} + \frac{M_{1,t-1}-M_{1,t}}{p_t} + v_1(\bar{L}_1 + L_{1,t})\right)$$

subject to for all t

$$(7) \qquad -\bar{L}_1 \leqslant L_{1,t} \leqslant \frac{M_{1,t-1}}{w_t}$$

$$(8) \qquad 0 \leqslant M_{1,t} \leqslant M_{1,t-1} - w_t L_{1,t}$$

$$(9) \qquad M_{1,0} = 0.$$

Similarly, the reduced form problem (P_2') of each producer is

$$(P_2') \qquad \max_{\{L_{2,t}\},\{M_{2,t}\}} \sum_{t=1}^{2} \beta_2^{t-1}\, U_2\left(f_2(L_{2,t}) - \frac{w_t}{p_t}L_{2,t} + \frac{M_{2,t-1}+X_{2,t}-M_{2,t}}{p_t}\right)$$

subject to for all t

$$(10) \qquad 0 \leqslant L_{2,t} \leqslant \frac{M_{2,t-1}+X_{2,t}}{w_t}$$

$$(11) \qquad 0 \leqslant M_{2,t} \leqslant M_{2,t-1} + X_{2,t} - w_t L_{2,t} + p_t f_2(L_{2,t})$$

$$(12) \qquad M_{2,2} \geqslant \tau_2$$

$$(13) \qquad M_{2,0} = M_0/N_2.$$

Proposition 1: *Stationary monetary competitive equilibrium* $\{p_t^*, w_t^*, c_{i,t}^*, L_{i,t}^*, q_{i,t}^*, M_{i,t}^* \,|\, i = 1,2\}_{t=1}^2$ *of a financially constrained economy* $\mathcal{FCE}$

(i) exists if and only if the following condition is satisfied:

$$\textbf{C1.} \quad 1+\alpha \geqslant \max\left\{\beta_1, \frac{\beta_2 f_2'(L_2^*)L_2^*}{f_2(L_2^*)}\right\}$$

where L_2^* *is equilibrium labor demand satisfying*

$$\frac{\beta_2}{1+\alpha} f_2'(L_2^*) = v_1'(\bar{L}_1 - (N_2/N_1)L_2^*);$$

(ii) is uniquely characterized by (14)-(21) for all t:

$$w_t^* = \frac{M_t}{N_2 L_{2,t}^*} \tag{14}$$

$$\frac{w_t^*}{p_t^*} = v_1'(\bar{L}_1 + L_{1,t}^*) \tag{15}$$

$$\frac{w_t^*}{p_t^*} = \frac{\beta_2}{1+\alpha} f_2'(L_{2,t}^*) \tag{16}$$

$$L_{2,t}^* = -\frac{N_1}{N_2} L_{1,t}^* \tag{17}$$

$$q_{i,t}^* = -\frac{w_t^*}{p_t^*} L_{i,t}^*, \qquad i = 1, 2 \tag{18}$$

$$c_{i,t}^* = f_i(\bar{L}_i + L_{i,t}^*) + q_{i,t}^*, \qquad i = 1, 2 \tag{19}$$

$$M_{1,t}^* = 0 \tag{20}$$

$$M_{2,t}^* = \frac{M_t}{N_2} \tag{21}$$

PROOF: See Appendix.

3.2. Intuition for Proposition 1

Due to increasing returns to scale, producers find it optimal to spend their whole cash in the labor market at the beginning of each period. By equal treatment property of the solution, producers equally share the total labor supply, and nominal wage is determined as the liquidity per unit of labor supplied.

In equilibrium, workers do not transfer money across successive periods. Workers spend their wage earnings entirely in the good market in each period. On the opposite side, producers turn out to own the total stock of money. However, even their motive for holding money cannot be explained by money hoarding. Producers need cash as a working capital to pay wage receipts in the labor market at the beginning of each period. Even if a producer (nonoptimally) chose not to produce in a particular period, he would definitely need cash for his good market purchases so as to make positive consumption as his life-time optimal plan suggests. Besides, producers need sufficient money balances at the end of the last period of their lives to pay their terminal taxes to the government. Money, for both workers and producers, mainly serves as a medium of exchange. Moreover, money has a restricted store of value for workers within periods and for producers across periods, but it is not (enthusiastically) hoarded by either of the parties.

It is striking that cash-in-advance constraints on labor purchases prevent an infinite excess demand and a due disequilibrium in the labor market that would otherwise arise in the presence of IRTS technologies. A single worker's labor supply, by equation (15), is given by

$$w_t/p_t = v_1'(\bar{L}_1 + L_{1,t}).$$

There is a trade-off, faced by workers, in choosing the optimal amount of labor to supply since more supply means higher wage revenue while at the same time less leisure. At the real wage rate w_t/p_t, the trade-off is just balanced. The concavity of v assures that the amount of labor supplied, which is $-L_{1,t}$, is increasing in the real wage rate.

On the other side, each producer has a labor demand satisfying

$$w_t/p_t = \frac{\beta_2}{1+\alpha} f_2'(L_{2,t}),$$

as shown in (16). The intuition underlying this equality is that increasing consumption at time t by reducing savings for the next period by ΔM units yields to a producer a marginal utility of $U_2'(c_t)(\Delta M/p_t)$, where $\Delta M/p_t$ is the amount of additional consumption. On the other hand, with ΔM units of reduction in period $t+1$ money holdings, the labor demand of the producer falls by $\Delta M/w_{t+1}$ units, which implies a reduction of $f_2'(L_{2,t+1})\Delta M/w_{t+1}$

units in output. The decrease in the utility of producer due to the fall in output at time $t+1$ then becomes $\beta_2 U_2'(c_{2,t+1}) f_2'(L_{2,t+1}) \Delta M / w_{t+1}$. In equilibrium, where consumption and effective labor are stationary and all money prices grow at the money inflation rate, the net marginal utility of transferring money from period t to period $t+1$ is zero only if $w_{t+1} = (1+\alpha) w_t = \beta_2 f_2'(L_{2,t+1}) p_t$.

It is to be seen that real wage rate is below the marginal product of labor for all feasible money growth rates exceeding β_2. This result must not be reckoned for the presence of increasing returns. The same productivity-wage gap also arises in cash-in-advance models with decreasing or constant return to scale technologies.[8] This gap is eliminated only if the rate of money inflation equals the time preference, β_2, of producers. Such a money supply rule, however, is not feasible as it would force consumption to be negative.

In general, the stationary equilibrium breaks down whenever the money inflation is not sufficiently high (in that case condition $C1$ in Proposition 1-i is violated). If $1+\alpha < \beta_1$, workers would desire to increase their end of first period money balances, so a stationary plan with zero money holding cannot be optimal. If $1+\alpha < \beta_2 f_2'(L_2^*) L_2^* / f_2(L_2^*)$ where L_2^* is SMCE level of employment, then the residual consumption of producers $f_2(L_2^*) - w_t^* L_2^* / p_t^*$ becomes negative, contradicting the optimality of the equilibrium.

3.3. Intuition for the Restrictions in the Model

The proof of Proposition 1 explains the need and use of various assumptions in the model such as that the equilibrium concept relates to the notion of stationarity, agents live for two periods, and money is tax backed. All such restrictions exist because of technical difficulties in solving agents' problems in the presence of IRTS technologies. In fact, none of the above assumptions are needed to characterize equilibrium under decreasing returns to scale (DRTS) technologies.[9] Money market equilibrium always requires

[8] See, for example, Fuerst (1992), Basci and Saglam (1999, 2000).

[9] See Basci and Saglam (2000).

money be held by at least one of the types. The concavity of a DRTS production function altogether with the concavity of utility function would suffice for the optimality of a unique interior solution (positive money holding and consumption in each period) to the optimization problem of a producer. But, in the presence of IRTS technologies, the objective functions of producers are no longer concave, and no sufficiency theorem is known to derive the desired result. The proof of optimality for this case involves an additional step that from the viewpoint of producers corner solutions with zero consumption at some of the periods are inferior to interior solution(s).

We should also note that it is the assumption of stationarity that makes an explicit and simple characterization of the equilibrium possible. That the model ends after two periods is an assumption made to show the uniqueness of the interior solution to the producers' problem. Finally, why money is tax backed should be obvious from the analysis of any monetary economy of finite horizon. If no money taxes were charged at the end of the last period, producers would try to make their end-of-life money balances go to zero. This is possible only if producers decline to sell their output in the good market. But, then, there would be no incentive for workers to supply labor in the last period for a wage earning that they can never spend. A backward induction shows that money would, indeed, be worthless in every period. The operation of a finite horizon economy through fiat money can be restored by introducing a positive money tax at the end of the last period. The level of such a terminal tax will, by the above arguments, affect the level of money demand in each period. Therefore, a stationary equilibrium is possible only if the money injected to the economy throughout the lives of the agents is totally taxed in the end.

4. Analysis of Equilibrium

A graphical exposition of the equilibrium is employed to simplify the analysis. Figure 1 depicts workers' total labor supply L_t^s and producers' total labor demand L_t^d in period t as a function of the real wage rate w_t/p_t. (Here, we assume $v_1''' > 0$ and $f_2''' < 0$ for a convex labor supply function and concave labor demand function, respectively.)

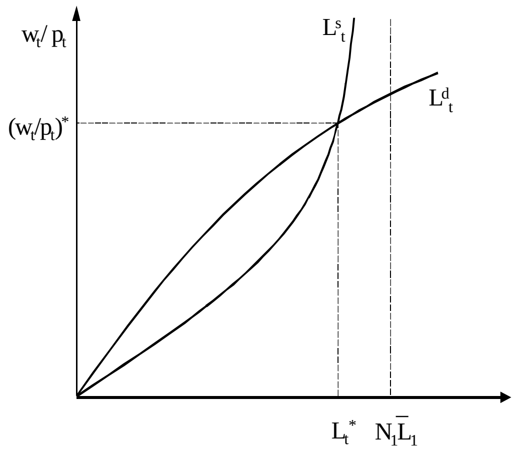

Figure 1:

Noting that $L_t^s = -N_1 L_{1,t}$ and $L_t^d = N_2 L_{2,t}$, we have labor market supply function

$$\frac{w_t}{p_t} = v_1' \left(\bar{L}_1 - \frac{L_t^s}{N_1} \right)$$

and labor market demand function

$$\frac{w_t}{p_t} = \frac{\beta_2}{1+\alpha} f_2' \left(\frac{L_t^d}{N_2} \right).$$

Here, the schedule L_t^d can be interpreted as the optimal labor demand schedule of producers implied by Euler conditions. It is due to increasing returns in production that L_t^d is upward sloping. The intersection of labor demand and labor supply schedules yields the equilibrium wage rate $(w_t/p_t)^*$ and employment level L_t^*.

Effective (liquidity-constrained) demand schedule is represented by $L_t^d = M_t/w_t$ that is drawn in Figure 2. Given the equilibrium employment level L_t^*, equilibrium nominal wage rate w_t^* is uniquely determined. One should notice that at the wage rate w_t^*, effective labor demand and optimal labor demand just match.

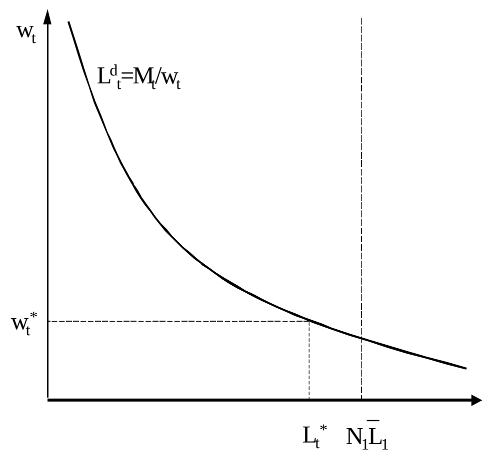

Figure 2:

Aggregate demand and aggregate supply schedules at the stationary money holding plan are drawn in Figure 3. The total demand for the good by workers at a stationary money holding is M_t/p_t and represented by the negatively sloped $AD_{1,t}$ curve. The aggregate demand for the good by workers and producers is represented by $AD_{1+2,t}$ curve, which always coincides with the vertical aggregate supply curve AS_t in the price-output plane, since producers are claimants of the residual demand.

It is apparent that AS-AD schedules are incapable to determine the equilibrium price, p_t^*, of the good. At the price p_t^*, which is uniquely determined once $(w_t/p_t)^*$ and w_t^* have been done so, workers demand $AD_{1,t}(p_t^*)$, which falls short of the total supply $AS_t(p_t^*) = N_2 f_2(L_t^*/N_2)$. The residual demand

$AS_t(p_t^*) - AD_{1,t}(p_t^*)$ is then consumed by producers. Note that the average cost pricing equilibrium that will be discussed in the next section corresponds to the price p_t^a, which is, in general, lower than the SMCE price.

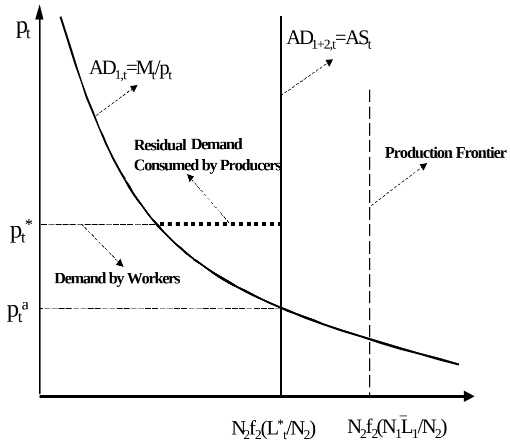

Figure 3:

Our first observation is that the economy is always at full employment, i.e. all the unemployment is voluntary. Like in the classical model, supply creates its own demand. However, in the presence of non-conventional labor demand function that we have derived, it is still fruitful to raise the question as to how changes in the parameters of the society and monetary policy affect equilibrium outcome and wealth distribution.

To relate first the technological change to the above question, assume an IRTS production function characterized by $f(L) = \theta\tilde{f}(L)$, where $\theta > 0$ represents the technology level. The marginal product of labor, hence the equilibrium real wage rate, are linear in θ. Thus an increase in θ reduces labor demand at each real wage rate, i.e. it shifts up the L_t^d schedule in Figure 1, leading to an increase in the equilibrium real wage rate $(w_t/p_t)^*$ and employment level L_t^*. Since the liquidity in the market has remained constant, it must be true from Figure 2 that the equilibrium nominal wage rate w_t^* has decreased. The increase in $(w_t/p_t)^*$ altogether with the decrease in w_t^* implies a decrease in the apple price p_t^*. With a higher level of employment, aggregate output and consumption increase. Aggregate supply curve in Figure 3 moves towards the production frontier. Workers now consume more. However, the net effect on the consumption of producers is ambiguous.

On the other side, an increase in total labor supply due to a rise in the number of workers N_1 or per worker labor supply $\bar{L}_1$ makes the labor supply schedule in Figure 1 shift to the right, leading, in equilibrium, to a rise in the real wage rate and employment, a fall in the nominal wage rate and in the apple price. Here, not only the aggregate supply curve but also the production possibility frontier move to the right in Figure 3. The wealth effects on workers and producers are qualitatively the same as in the formerly discussed case of a positive technology shock.

We also observe that the patience of producers affects the distribution of wealth in the society. An increase in the time preference β_2 of producers shifts up the labor demand schedule in Figure 1, thereby raising the equilibrium real wage rate, employment, and output. The wealth effects are similar to those discuseed in the previous cases.

Now consider a change in the competitiveness of the production sector, say due to a rise in the number of producers. Total labor demanded by producers increases at each real wage rate, so L_t^d schedule in Figure 1 shifts to the right, which implies a fall in the equilibrium real wage rate, employment and per producer output. The nominal wage rate must now be higher due to binding liquidity constraint in factor payments and the reduced employment, which further implies higher apple price. Then workers consume less. Aggregate supply and demand are also reduced as apparent from the first-order differentiation of $AS_t = N_2 f_2(L_t^*/N_2)$ with respect to N_2. However, the net effect in residual consumption of producers is ambiguous. It turns out that although the competition reduces real wage rate, this is not without any social costs. The total output and employment decrease; this may even be true for producers profits.

We can now discuss the effect of the quantity and the growth of money on the equilibrium outcome. Since the level of money enters neither the labor supply curve nor labor demand curve, which are the structural determinants of the output, the quantity of money has no effect on the equilibrium real wage rate, employment and output. Besides, given a constant money supply rule, an increase in the initial money stock M_0 shifts up the liquidity-constrained labor demand schedule in each period as should be evident from Figure 2. This causes an equal increase (in percentage terms) in the equilibrium nominal wage rate for each period. Since the equilibrium real wage rate does not change, the equilibrium price of apples must increase at the same rate as the money supply does. As the relative prices and the total output remain constant, a money shock produces no wealth effects.

A final issue to consider here is whether money is superneutral, i.e. a change in the money growth rate has real effects. An increase in α causes the labor demand curve in Figure 1 to shift to the right. In effect, the equilibrium real wage rate, employment, and output decrease. The increase in the money stock in each period due to a higher money growth rate shifts up the effective labor demand schedule in Figure 2. Lower employment level now corresponds to a higher nominal wage rate. The percentage change in the nominal wage rate in each period is higher than that in the money growth rate. The fall in the real wage rate implies an increase in the apple price that is more dramatic (in percentage terms) than the increase in the nominal wage rate. This implies a reduction of the equilibrium real money balances M_t/p_t^*. Thus, even though aggregate demand curve of workers in Figure 3 shifts to the right in response to an increase in money stock for a higher growth rate, equilibrium price increase is so high that the total demand of workers decreases. The net effect on the residual demand of producers is ambiguous since the output in the economy has decreased, too.

It follows from the above discussion together with Proposition 1-i that the optimal monetary policy from the viewpoints of workers and the society as a whole must satisfy

$$1+\alpha = \max\left\{\beta_1, \frac{\beta_2 f_2'(L_2^*)L_2^*}{f_2(L_2^*)}\right\}.$$

It is very striking that when workers are 'very' impatient (the case of sufficiently low β_1), the optimal money growth rule becomes $1+\alpha = \beta_2 f_2'(L_2^*)L_2^*/f_2(L_2^*)$. In that case the equilibrium price $p_t^* = w_t^* L_2^*/f_2(L_2^*)$ coincides with the price p_t^a in Figure 3, which just equates workers' apple demand $w_t^* N_2 L_2^*/p_t^* = M_t/p_t^*$ to the total market supply $N_2 f_2(L_2^*)$. The economy is, then, at an average-cost-pricing equilibrium, in which producers obtain zero profits. One should here note that the marginal cost pricing that would be implied by the money growth rate $1+\alpha = \beta_2$ is not feasible since it causes, as explained in Section 3.2 , the equilibrium to break down.

It is also interesting to see that the optimal money supply rule is not always deflationary. For example, the optimal rule simply becomes

$$1+\alpha = \max\left\{\beta_1, \beta_2\eta\right\}$$

for the class of production functions given by $f(L) = \theta L^\eta$. Then for $\eta > \beta_2^{-1}$, the optimal policy is to inflate money at the rate $\alpha = \beta_2\eta - 1 > 0$.

5. CONCLUDING REMARKS

In this paper, we show that a price taking equilibrium may prevail even under increasing returns if money is needed for production. This financing need is assumed to be met by owner's equity capital and retained earnings. External financing from a competitive loan market mostly assumed in the related business cycles literature (Fuerst (1992), Christiano et al. (1997, 1998, 2001)) cannot operate here since the demand for loans would be unbounded for any given level of interest rates, regardless of how high they may be.

The existence proof, although indirect and tedious, gives rise as a by product to uniqueness and allows the direct use of Euler equations to characterize the equilibrium, provided that the stated extra assumptions are met. Moreover, the main result of the paper characterizes the equilibrium in a way to allow back of the envelope of graphical analysis of equilibrium.

The analysis reveals some non-conventional results. The first important result is that reducing the number of firms, without touching the competitive price taking assumption, gives rise to a higher output and welfare.

The second important result is the lower bound on inflation and money growth for the existence of a steady state competitive equilibrium. This lower bound is monotonically increasing with the extent of returns to scale. Therefore the monetary policy of low inflation should be applied with an eye on this lower limit if this class of models have any relevance in practice.

APPENDIX

Lemma A.1. Let $h : \mathbb{R}_{++} \to \mathbb{R}_{++}$ be a function that is strictly increasing (decreasing) and multiplicatively separable. Define $g := 1/h$. Then the inverse function g^{-1} is strictly decreasing (increasing) and separable.

Proof: Both g and g^{-1} are strictly decreasing (increasing), since h is strictly increasing (decreasing). It is also obvious that g is separable, since h is separable. Now pick any $x_1, x_2 \in \mathbb{R}_{++}$. Suppose

$$g^{-1}(x_1 x_2) \neq g^{-1}(x_1) g^{-1}(x_2).$$

Then the monotonicity and separability of g implies

$$g(g^{-1}(x_1 x_2)) \neq g(g^{-1}(x_1) g^{-1}(x_2))$$

or

$$x_1 x_2 \neq g(g^{-1}(x_1)) g(g^{-1}(x_2)) = x_1 x_2$$

which is a contradiction. Thus g^{-1} is separable. *Q.E.D.*

PROOF OF PROPOSITION 1: We will consider the two parts of the proposition separately.

Part i: To prove the 'only if' part, consider the objective of workers as expressed in (P_1'):

$$\max_{\{L_{1,t}\},\{M_{1,t}\}} \sum_{t=1}^{2} \beta_1^{t-1} U_1 \left(-\frac{w_t}{p_t} L_{1,t} + \frac{M_{1,t-1} - M_{1,t}}{p_t} + v_1(\bar{L}_1 + L_{1,t}) \right)$$

The first-order condition for $L_{1,t}$ yields the labor supply curve $w_t/p_t = v_1'(\bar{L}_1 + L_{1,t})$. The cash-in-advance constraint is not binding for workers.

Note that if $1+\alpha \geqslant \beta_1$, the Euler condition at any stationary plan and prices, associated with the control $M_{1,1}$, becomes

$$-\frac{1}{p_1} U_1'(c_{1,1} + v_1(\bar{L}_1 + L_{1,1})) + \frac{\beta_1}{p_2} U_1'(c_{1,2} + v_1(\bar{L}_1 + L_{1,2})) \leqslant 0,$$

since $c_{1,2} = c_{1,1}$, $L_{1,2} = L_{1,1}$ and $p_2 = p_1(1+\alpha)$. So, the stationary money holding plan $M_{1,2} = M_{1,1} = M_{1,0} = 0$ is optimal if $1+\alpha \geqslant \beta_1$.

If, on the contrary, $1+\alpha < \beta_1$, the Euler condition, for stationary prices and consumption, becomes

$$-\frac{1}{p_1} U_1'(c_{1,1} + v_1(\bar{L}_1 + L_{1,1})) + \frac{\beta_1}{p_2} U_1'(c_{1,2} + v_1(\bar{L}_1 + L_{1,2})) > 0.$$

Then, by slightly increasing $M_{1,1}$ over $M_{1,0} = 0$ (hence slightly increasing $c_{1,2}$ over $c_{1,1}$) workers can be better off. In that case, the stationary plan $M_{1,t} = 0$ would not be optimal.

To show that equilibrium exists only if $1+\alpha \geqslant \beta_2 f_2'(L_2^*) L_2^* / f_2(L_2^*)$ is reserved to Part ii-(a) of the proof.

Finally, the proof of the 'if' statement in Part-i of Proposition 1 is implicit in the proof of Part-ii-(b).

Part ii: Let $M_{1,0} = 0$. The rest of the proof consists of two parts: (a) Every SMCE satisfies (14)-(21) for all t; (b) the plan (14)-(21) is a SMCE.

(a) Let $\{w_t, p_t, L_{i,t}, q_{i,t},\ c_{i,t}, M_{i,t} \,|\, i = 1, 2\}_{t=1}^{2}$ be a SMCE. In part (i) of the proof, we showed that the real wage rate and labor supply of each worker must satisfy (15). Labor market clearing implies (17). Equation (18) follows from (4) at a stationary money holding plan, while (19) is a restatement of (1) in equilibrium. Stationarity of money balances imply (20) and (21).

To derive the rest of the SMCE plan, consider the objective of type 2 agents in the reduced problem (P_2'):

$$\max_{\{L_{2,t}\},\{M_{2,t}\}} \sum_{t=1}^{2} \beta_2^{t-1}\, U_2\left(f_2(L_{2,t}) - \frac{w_t}{p_t} L_{2,t} + \frac{M_{2,t-1} + X_{2,t} - M_{2,t}}{p_t}\right)$$

The first-order condition for $L_{2,t}$, under the fact that f_2 has increasing returns to scale, implies $L_{2,t} = (M_{2,t-1} + X_{2,t})/w_t$. That is cash-in-advance constraint is binding for type 2 agents. (The assumption that $1+\alpha > \beta_2 f_2'(L_2^*)L_2^*/f_2(L_2^*)$ guarantees nonnegative profits to each producer, thus we suppose that the other corner solution $L_{2,t} = -\bar{L}_2 = 0$ is not chosen.) Equation (14), thus, follows from (21) and $L_{2,t} = M_{2,t}/w_t$.

The objective of type 2 agents, then, reduces to

$$\max_{\{M_{2,t}\}} \sum_{t=1}^{2} \beta_2^{t-1}\, U_2\left(f_2\left(\frac{M_{2,t-1} + X_{2,t}}{w_t}\right) - \frac{M_{2,t}}{p_t}\right).$$

The Euler condition at a stationary plan and prices, associated with the control $M_{2,1}$, becomes

$$-\frac{1}{p_1} U_2'(c_{2,1}) + \frac{\beta_2 f_2'(L_{2,2})}{w_2} U_2'(c_{2,2}) = 0.$$

From the stationarity conditions, $c_{2,1} = c_{2,2}$, $w_2 = (1+\alpha)w_1$, and $L_{2,1} = L_{2,2}$ it follows that

$$w_t/p_t = \beta_2 f_2'(L_{2,t})/(1+\alpha).$$

Together with (14) and (15), one then obtains the equilibrium price of apples.

At SMCE plan and prices, the consumption of each producer, given by

$$c_{2,t} = f_2(L_{2,t}) - \frac{w_t}{p_t} L_{2,t}$$

is positive only if

$$1 + \alpha \geqslant \beta_2 f_2'(L_{2,t}) L_{2,t} / f_2(L_{2,t}).$$

(b) We have to prove that the plan (14)-(21) is optimal, individually feasible, stationary, and satisfies aggregate feasibility (market clearing) conditions.

(b-i) Optimality: We will check that both types of agents optimize under the proposed prices and plans of action. The optimality of $L^*_{1,t}$ and $L^*_{2,t}$ were shown in part i of the proof. It is left to prove that given $M_{1,0} = 0$ and $M_{2,0} = M_0/N_2$, the SMCE money holding plans $M^*_{1,t}$ and $M^*_{2,t}$ are respectively optimal for type 1 and 2 agents under the stationary prices and wages $\{p^*_t, w^*_t\}^2_{t=1}$. For ease of notation, suppress the superscript $(*)$ in equilibrium prices and wages, hereafter.

Denote the objective function of type i agents in (P_i') as $V_i(M_{1,1}, M_{1,2})$. Define $V_i^t(M_{i,1}, M_{i,2}) := \partial V_i(M_{i,1}, M_{i,2})/\partial M_{i,t}$ for $t = 1, 2$. First consider

$$\begin{aligned} V_1(M_{1,1}, M_{1,2}) &= U_1\left(\frac{w_1}{p_1} L_{1,1} - \frac{M_{1,1}}{p_1} + v_1(\bar{L}_1 - L_{1,1})\right) \\ &\quad + \beta_1 U_1\left(\frac{w_2}{p_2} L_{1,2} + \frac{M_{1,1} - M_{1,2}}{p_2} + v_1(\bar{L}_1 - L_{1,2})\right). \end{aligned}$$

Then

$$V_1^1 = -\frac{1}{p_1} U_1'\left(c_{1,1} + v_1(\bar{L}_1 - L_{1,1})\right) + \frac{\beta_1}{p_2} U_1'\left(c_{1,2} + v_1(\bar{L}_1 - L_{1,2})\right) \leqslant 0,$$

$$V_1^2 = -\frac{\beta_1}{p_2} U_1'\left(c_{1,2} + v_1(\bar{L}_1 - L_{1,2})\right) < 0,$$

and

$$V_1^{1,1} = \frac{1}{p_1^2} U_1'' \left(c_{1,1} + v_1(\bar{L}_1 - L_{1,1})\right) + \frac{\beta_1}{p_2^2} U_1'' \left(c_{1,2} + v_1(\bar{L}_1 - L_{1,2})\right) < 0,$$

$$V_1^{2,2} = \frac{\beta_1}{p_2^2} U_1'' \left(c_{1,2} + v_1(\bar{L}_1 - L_{1,2})\right) < 0.$$

Since $V_1^2(M_{1,1}, 0) < 0$, $M_{1,2} = 0$. Moreover, $V_1^{1,1}(0,0) < 0$. So, $M_{1,1} = 0$. Therefore, $c_{1,t} = w_t L_{1,t}/p_t$ for all t.

At SMCE prices the objective function of type 2 agent is

$$\begin{aligned} V_2(M_{2,1}, M_{2,2}) &= U_2 \left(f_2 \left(\frac{M_{2,0} + X_{2,1}}{w_1} \right) - \frac{M_{2,1}}{p_1} \right) \\ &\quad + \beta_2 U_2 \left(f_2 \left(\frac{M_{2,1} + X_{2,2}}{w_2} \right) - \frac{M_{2,2}}{p_2} \right) \end{aligned}$$

Type 2 agents must satisfy $M_{2,2} \geqslant \tau_2$. However,

$$V_2^2(M_{2,1}, M_{2,2}) = -\frac{\beta_2}{p_2} U_2' (c_{2,2}) < 0$$

since $c_{2,2} = f_2((M_{2,1} + X_{2,2})/w_2) - M_{2,2}/p_2) \geqslant 0$ and

$$V_2^{2,2}(M_{2,1}, M_{2,2}) = \frac{\beta_2}{p_2^2} U_2'' (c_{2,2}) < 0.$$

Thus, optimality requires $M_{2,2} = \tau_2$.

Define $m_{2,t-1} := M_{2,t-1}/w_t$ for $t = 1, 2, 3$, and $x_{2,t} := X_{2,t}/w_t$ for $t = 1, 2$, where $w_3 = (1 + \alpha) w_2$ is an auxiliary wage rate.

It then follows that $V_2(M_{2,1}, M_{2,2}) = V_2(m_{2,1}w_2, m_{2,2}w_3)$. Noting the optimality condition $m_{2,2} = \tau_2/w_3$, we re-express $V_2(m_{2,1}w_2, m_{2,2}w_3)$ as $V_2(m_{2,1}w_2)$, or shortly $V_2(m_{2,1})$, where

$$V_2(m_{2,1}) = U_2 \left(f_2(m_{2,0} + x_{2,1}) - \frac{w_2}{p_1} m_{2,1} \right) + \beta_2 U_2 \left(f_2(m_{2,1} + x_{2,2}) - \frac{\tau_2}{p_2} \right).$$

Then we have

$$\begin{aligned} V_2'(m_{2,1}) &= -\frac{w_2}{p_1}U_2'\left(f_2(m_{2,0}+x_{2,1})-\frac{w_2}{p_1}m_{2,1}\right) \\ &\quad +\beta_2 f_2'(m_{2,1}+x_{2,2})U_2'\left(f_2(m_{2,1}+x_{2,2})-\frac{\tau_2}{p_2}\right). \end{aligned}$$

The rest of the proof aims to show the optimality of SMCE plan for producers, and involves three steps: First, we will show that there exists an interior solution to the first-order condition for the problem of producers for all money growth rates satisfying $1+\alpha > \beta_2 f_2'(L_2^*)L_2^*/f_2(L_2^*)$. Second, we will prove that corner solutions cannot be optimal and hence the optimal solution must be an interior one. The last step is to show that the interior solution is unique.

Step 1: Note that

$$m_{2,1} \to f_2^{-1}(\tau_2/p_2) - x_{2,2} \ \text{ implies } \ V_2' \to \infty, \text{ and}$$

$$m_{2,1} \to p_1 f_2(m_{2,0}+x_{2,1})/w_2 \ \text{ implies } \ V_2' \to -\infty.$$

Now denote the set of feasible money holding plans by $\mathcal{A} = [f_2^{-1}(\tau_2/p_2) - x_{2,2}, p_1 f_2(m_{2,0}+x_{2,1})/w_2)]$ and its interior by $int(\mathcal{A})$. One can easily show that $int(\mathcal{A}) \neq \emptyset$ if and only if

$$f_2^{-1}\Big(\frac{\beta_2 f_2'(L_2^*)L_2^*}{1+\alpha}\Big) - \frac{\alpha}{1+\alpha}L_2^* < \frac{f_2(L_2^*)}{\beta_2 f_2'(L_2^*)},$$

which holds true if $1+\alpha > \beta_2 f_2'(L_2^*)L_2^*/f_2(L_2^*)$. (Note that $int(\mathcal{A}) = \emptyset$ if $1+\alpha = \beta_2 f_2'(L_2^*)L_2^*/f_2(L_2^*)$. Then $c_{2,1} = c_{2,2} = 0$ and producers cannot do anything better than following the SMCE plan.)

Since V_2' is continuous in $m_{2,1}$, there exists $m_{2,1} \in int(\mathcal{A})$ such that $V_2'(m_{2,1}) = 0$.

Step 2: First consider the consumption plan

$$c_{2,1} = f_2(m_{2,0}+x_{2,1}) - (w_2/p_1)[f_2^{-1}(\tau_2/p_2) - x_{2,2}] \ \text{ and } \ c_{2,2} = 0,$$

associated with the corner solution $m_{2,1} = f_2^{-1}(\tau_2/p_2) - x_{2,2}$. Assumption $A1$ ensures that producers have an incentive to slightly increase $m_{2,1}$ and hence $c_{2,2}$, if $int(\mathcal{A}) \neq \emptyset$.

Next consider the consumption plan

$$c_{2,1} = 0 \quad \text{and} \quad c_{2,2} = f_2(p_1 f_2(m_{2,0} + x_{2,1})/w_2) - \tau_2/p_2,$$

associated with the corner solution $m_{2,1} = p_1 f_2(m_{2,0} + x_{2,1})/w_2$. Assumption $A1$ now ensures that slightly decreasing $m_{2,1}$, hence increasing $c_{2,1}$, makes producers better-off if $int(\mathcal{A}) \neq \emptyset$. Therefore, no corner solution can be optimal.

Step 3: At an interior solution, the Euler condition becomes

$$\begin{aligned}&U_2'\left(f_2(m_{2,0} + x_{2,1}) - \frac{w_2}{p_1} m_{2,1}\right) = \\ &\frac{\beta_2}{w_2/p_1} f_2'(m_{2,1} + x_{2,2}) U_2'\left(f_2(m_{2,1} + x_{2,2}) - \frac{\tau_2}{p_2}\right).\end{aligned}$$

Define $g := (1/U_2')^{-1}$. The function U_2' is strictly decreasing and separable, by assumption. Thus, by Lemma A.1., g is strictly increasing and separable. Then

$$\begin{aligned}&f_2(m_{2,1} + x_{2,2}) - \tau_2/p_2 = \\ &g\left(\frac{\beta_2}{w_2/p_1} f_2'(m_{2,1} + x_{2,2}) \frac{1}{U_2'(f_2(m_{2,0} + x_{2,1}) - m_{2,1}\, w_2/p_1)}\right).\end{aligned}$$

Since g is separable,

$$\begin{aligned}&f_2(m_{2,1} + x_{2,2}) - \tau_2/p_2 = \\ &g(\beta_2 p_1/w_2) g(f_2'(m_{2,1} + x_{2,2}))[f_2(m_{2,0} + x_{2,1}) - m_{2,1}\, w_2/p_1].\end{aligned}$$

Then

$$\begin{aligned}\tau_2/p_2 \;&=\; f_2(m_{2,1} + x_{2,2}) \\ &\quad - g(\beta_2 p_1/w_2) g(f_2'(m_{2,1} + x_{2,2}))[f_2(m_{2,0} + x_{2,1}) - m_{2,1}\, w_2/p_1].\end{aligned}$$

Denoting RHS of the above equation by $h(m_{2,0}, m_{2,1})$, we have

$$\tau_2/p_2 = h(m_{2,0}, m_{2,1}),$$

and $\tau_2 > 0$ is feasible only if $h(m_{2,0}, m_{2,1}) > 0$, that is

$$\frac{f_2(m_{2,1} + x_{2,2})}{g(\beta_2 p_1/w_2) g(f_2'(m_{2,1} + x_{2,2}))} + m_{2,1}\, w_2/p_1 > f_2(m_{2,0} + x_{2,1}).$$

If we can show that $h_2 := \partial h(m_{2,0}, m_{2,1})/\partial m_{2,1} > 0$ for all $m_{2,1}$ satisfying $h(m_{2,0}, m_{2,1}) > 0$ for a given $m_{2,0}$, then given any $m_{2,0}$ there exists a unique solution for $m_{2,1}$ to satisfy the Euler equation. Note that

$$\begin{aligned} h_2 &= f_2'(m_{2,1} + x_{2,2}) \\ &\quad - g(\beta_2 p_1/w_2) g'(f_2'(m_{2,1} + x_{2,2})) f_2''(m_{2,1} + x_{2,2}) [f_2(m_{2,0} + x_{2,1}) \\ &\quad - m_{2,1}\, w_2/p_1] + g(\beta_2 p_1/w_2) g(f_2'(m_{2,1} + x_{2,2}))(w_2/p_1). \end{aligned}$$

We have $h_2 > 0$ if

$$\begin{aligned} &\frac{f_2'(m_{2,1} + x_{2,2})}{g(\beta_2 p_1/w_2) g'(f_2'(m_{2,1} + x_{2,2})) f_2''(m_{2,1} + x_{2,2})} \\ &+ \frac{g(f_2'(m_{2,1} + x_{2,2}))\, w_2/p_1}{g'(f_2'(m_{2,1} + x_{2,2})) f_2''(m_{2,1} + x_{2,2})} + m_{2,1}\, w_2/p_1 > f_2(m_{2,0} + x_{2,1}). \end{aligned}$$

By assumption $A3$, $f_2(m_{2,1}+x_{2,2})/g(f_2'(m_{2,1}+x_{2,2}))$ is increasing in $m_{2,1}+x_{2,2}$, hence in $m_{2,1}$. So

$$\frac{f_2(m_{2,1} + x_{2,2})}{g(f_2'(m_{2,1} + x_{2,2}))} < \frac{f_2'(m_{2,1} + x_{2,2})}{g'(f_2'(m_{2,1} + x_{2,2})) f_2''(m_{2,1} + x_{2,2})},$$

implying $h_2 > 0$.

(b-ii) Individual feasibility: In equilibrium, condition (7) of (P_1') is satisfied in the interior. At SMCE, the set of inequalities in (8) of (P_1') reduces to $0 \leqslant -w_t L_{1,t}$; which, indeed, holds true.

For type 2 agents, condition (10) holds at the boundary as $L_{2,t} = (M_{2,t-1} + X_{2,t})/w_t$ at SMCE. Condition (11) of (P_2') reduces to $0 \leqslant -w_t L_{2,t} + p_t f_2(L_{2,t})$.

The RHS of this inequality is $p_t c_{2,t}$, the market value of consumption at period t, which is positive, by the nonoptimality of the corner solution. Finally, condition (12) of (P_2') holds with equality sign in equilibrium.

(b-iii) Aggregate feasibility: Equation (17) is consistent with labor market clearing. The plans (18) clear the good market, and money holding plans (20) and (21) are consistent with the money market clearing.

(b-iv) Stationarity: Obviously, the plan (14)-(21) is stationary. *Q.E.D.*

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